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A linearization of a backward Euler scheme for a class of degenerate nonlinear advection–diffusion equations☆

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Abstract

In this paper we propose a linearization of a backward scheme for a class of degenerate nonlinear advection–diffusion equations, modeling a two-phase flow through a porous medium. This is done using a Taylor expansion of the fractional flow function, and of the inverse function of an antiderivative of the diffusion coefficient. The coercivity of the matrix obtained in the linearization is shown for a sufficiently small time step.

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1. Introduction

We consider the nonlinear convection–diffusion problem [7,5,8,11–13] obtained by modeling a two-phase flow through a porous medium [1,4,9,3]

$$\begin{cases} \frac{\partial}{\partial t} S + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S) \quad \text{on } \Omega \times (0, T_0], \\ (f(S)\mathbf{u} - k(S)\nabla S) \cdot \mathbf{n} = q \quad \text{on } \partial\Omega \times [0, T_0], \\ S(x, 0) = S^0(x) \quad \text{on } \Omega \end{cases} \tag{1.1}$$

with Ω a bounded domain of $\mathbf{R}^n$, $n = 1, 2, 3$, with smooth boundary $\partial\Omega$, and where $0 \leqslant S^0(x) \leqslant 1$, for all $x \in \Omega$. For simplicity let $|\Omega| = 1$.

The vector $\mathbf{n}$ is the outward unit normal to the boundary of Ω, S the saturation (for two-phase flow) of the invading phase, f the fractional flow function, $\mathbf{u}$ the Darcy velocity, and k the conductivity of medium (a function of the permeabilities of the phases and the viscosity of the medium). We assume that $\mathbf{u}$ is sufficiently regular for the purpose of this analysis.

We encounter here two main problems:

(1) the nonlinearity of the problem,
(2) the degeneracies: k vanishes at 2 values of S, say, $S = 0$ and $S = 1$.

In [7,5,8], this problem was considered, and error estimates were established for a perturbation, a continuous Galerkin method, and a discrete Galerkin method. But the scheme considered for the discrete Galerkin method was nonlinear. A linearization in some way is necessary for the effective computation and visualization of the approximate solution. The most common linearization used is the Picard iteration (see [2,10], for instance). In this paper we propose a naive approach: linearize the coefficients of the equations through first-order Taylor expansions. This paper shows that the linear scheme obtained is well-defined. In a forthcoming paper, we will establish error estimates for this scheme.

In Section 2, we describe the usual successive approximations of the solution to problem (1.1). We show that the backward Euler scheme is well-defined. In Section 3 we state and prove the main result of this paper.

We make the following assumptions which are motivated and described in [7,8]:

$$k(0) = k(1) = 0,$$
$$k(\xi) \geqslant \begin{cases} c_1 \xi^{\mu} & \text{if } 0 \leqslant \xi \leqslant \alpha_1, \\ c_2 & \text{if } \alpha_1 < \xi < \alpha_2, \\ c_3(1-\xi)^{\mu} & \text{if } \alpha_2 \leqslant \xi \leqslant 1, \end{cases} \tag{1.2}$$

where $0 < \alpha_1 < \frac{1}{2} < \alpha_2 < 1$, and $0 < \mu \leqslant 2$.

Define

$$K(\xi) := \int_0^{\xi} k(\tau)\,\mathrm{d}\tau.$$

We also make the assumption:

$$C^* \| f(b) - f(a) \|^2 \leqslant (K(b) - K(a))(b - a) \tag{1.3}$$

which is reasonable since it holds under (1.2), if

$$f'(0) = f'(1) = 0. \tag{1.4}$$

We note that (1.4) is the usual assumption on the fractional flow function for the saturation equation. We also note that condition (1.3) implies the following.

$$f'(s) \leqslant C\sqrt{k(s)}. \tag{1.5}$$

Finally, we set additional notation which will be used throughout the remainder of this paper. We define $(f, g) := (f, g)_\Omega := \int_\Omega fg\,\mathrm{d}x$ when this has a meaning. The notation $\|f\|_{L^p} := \|f\|_{L^p(\Omega)}$ is used for the standard Lebesgue norm of a measurable function, when this quantity is finite. Similarly, we denote by $\|f\|_{L^p(L^q)} := \|f\|_{L^p(0,T,L^q(\Omega))}$ the mixed Lebesgue norm for f. We use C, c, to denote constants which may change from line, but which are independent of the parameters β, h and Δt, unless otherwise explicitly specified.

2. Preliminary

In this section we formulate the successive approximations of the problem: Regularization, discretization in space and full discretization. Because we do not need them here, we do not give any error estimates for any of this progression of approximations and refer instead to [5–8,12], for instance.

2.1. Regularized problem

Because of the degeneracies, the solution to the above problem is not guaranteed to be sufficiently smooth for numerical approximation purposes. Hence the problem is often regularized in some way [7,8,5,11–13]. Here we simply perturb the diffusion coefficient k. Replace k by k_β so that $k_\beta \to k$ strongly as the regularization parameter $\beta \to 0$.

For this purpose, let $0 < \beta \leqslant \frac{1}{2}$, and define k_β by

$$k_\beta(s) = \max(k(s), c_0\beta^\mu),$$

then $k_\beta(s) \geqslant \beta^\mu > 0$.

Another example of perturbation is as follows: Let

$$\delta = \min(k(\beta), k(1-\beta)) \tag{2.1}$$

and define k_β by

$$\begin{cases} k_\beta(s) = k(s) & \text{if } k(s) \geqslant \delta, \\ \frac{1}{2}\,\delta \leqslant k_\beta(s) \leqslant \delta & \text{otherwise,} \end{cases} \tag{2.2}$$

then $k_\beta(t) \geqslant k(t)$, for all $t \in [0, T]$, k_β satisfies

$$k_\beta(s) \geqslant \tfrac{1}{2}\delta \quad \forall s \in [0, 1], \tag{2.3}$$

and thus k_β is bounded away from 0.

Define

$$K_\beta(s) = \int_0^s k_\beta(\tau)\,\mathrm{d}\tau. \tag{2.4}$$

Replace k by k_β in the original problem to get the new, now nondegenerate, problem:

$$\begin{cases} \frac{\partial}{\partial t} S_\beta + \nabla \cdot (f(S_\beta)\mathbf{u}) - \nabla \cdot (k_\beta(S_\beta)\nabla S_\beta) = Q(S_\beta) & \text{on } \Omega \times (0, T_0], \\ (f(S_\beta)\mathbf{u} - k(S_\beta)\nabla S_\beta) \cdot \mathbf{n} = q \quad \text{on } \partial\Omega \times [0, T_0], \\ S_\beta(x, 0) = S^0(x) \quad \text{on } \Omega. \end{cases} \tag{2.5}$$

For the following analysis we assume, to simplify, that $Q \equiv 0$ and $q \equiv 0$.

2.2. *Continuous Galerkin method*

We now discretize in space. Let $\{M_h\}_{0<h<1}$ be a family of finite-dimensional spaces, with $M_h \subset H^1(\Omega)$. Extend k_β as follows (and call it again k_β):

$$k_\beta(\xi) = \begin{cases} k_\beta(\xi) & \text{if } \xi \geqslant 1, \\ k_\beta(-\xi) & \text{if } \xi \leqslant 0. \end{cases}$$

Then K_β is bijective from $\mathbf{R}$ to $\mathbf{R}$. Set

$$H_\beta = K_\beta^{-1}. \tag{2.6}$$

Consider the discretized problem

$$\left(\frac{\partial}{\partial t} H_\beta(V_h), \chi\right) - (f(H_\beta(V_h))\mathbf{u}, \nabla\chi) + (\nabla V_h, \nabla\chi) = 0 \tag{2.7}$$

for all $\chi \in M_h$, and $t \in (0, T_0]$ with the initial condition:

$$P_h H_\beta(V_h(0)) = P_h S^0, \tag{2.8}$$

where S^0 is as in (1.1), and P_h the L^2 projection on M_h. We are looking for V_h in M_h, where V_h is the Galerkin approximation to $K_\beta(S_\beta)$.

2.3. *The discrete Galerkin method*

Apply the backward Euler in time to the above semi-discretized scheme to get

$$\begin{aligned} &\left(\frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t}, \chi\right) - (f(H_\beta(V_h^{n+1}))\mathbf{u}^{n+1}, \nabla\chi) \\ &\quad + (\nabla V_h^{n+1}, \nabla\chi) = 0 \end{aligned} \tag{2.9}$$

for all $\chi \in M_h$, $n = 0, 1, \ldots, N-1$ with

$$P_h H_\beta V_h^0 = P_h S^0, \tag{2.10}$$

where $\mathbf{u}^n = \mathbf{u}(n\Delta t, x)$. We will also set the notation

$$\mathbf{u}' = \frac{\partial \mathbf{u}}{\partial t}.$$

As noted in [8], scheme (2.9)–(2.10) can be rewritten in the form

$$V_h^n = F V^{n+1} \quad n = 1, 2, \ldots, N-1 \tag{2.11}$$

for some nonlinear operator F. In [8] (see pages 373 and 374 of that paper), it was shown that this operator is well-defined (invertible), but the proof assumed that $\mathbf{u}$ was independent of time t. Here we restate a modified form of this fact for a general $\mathbf{u}$ and give a sketch of the proof for this case.

Theorem 2.1. *Let F be defined by* (2.11). *Let P_h be the L^2-projection onto M_h. Assume the above conditions on k and f hold. Then*

$$\begin{aligned}&(P_h H_\beta(F V_h^{n+1}) - P_h H_\beta(F V_h^{m+1}), V_h^{n+1} - V_h^{m+1})\\ &\quad \geqslant \left(1 - \Delta t \frac{\|\mathbf{u}\|^2_{L^\infty(L^\infty)}}{2C^*}\right) \|V_h^{n+1} - V_h^{m+1}\|^2_{L^2}\\ &\qquad + \frac{1}{2}\Delta t \|\nabla(V_h^{n+1} - V_h^{m+1})\|^2_{L^2}\\ &\qquad - C\Delta t \|f(\cdot)\|^2_{L^2} \|u'\|^2_{L^\infty(L^\infty)}.\end{aligned} \tag{2.12}$$

Proof. We give a sketch of the proof which goes as in [8], except that there is an additional term intervening. If V_h^n (resp. V_h^m) is the iterated solution at time $n\Delta t$ (resp. $m\Delta t$) to (2.9)–(2.10), then by (2.9) and (2.11), we have

$$\begin{aligned}&\left(\frac{P_h H_\beta(F V_h^{n+1}) - P_h H_\beta(F V_h^{m+1})}{\Delta t}, V_h^{n+1} - V_h^{m+1}\right)\\ &\quad + ((f(H_\beta(V_h^{n+1})) - f(H_\beta(V_h^{m+1})))\mathbf{u}^{m+1}, \nabla(V_h^{n+1} - V_h^{m+1}))\\ &\quad + (f(H_\beta(V_h^{n+1}))(\mathbf{u}^{n+1} - \mathbf{u}^{m+1}), \nabla(V_h^{n+1} - V_h^{m+1})\\ &\qquad = \left(\frac{P_h H_\beta(V_h^{n+1}) - P_h H_\beta(V_h^{m+1})}{\Delta t}, V_h^{n+1} - V_h^{m+1}\right)\\ &\quad + \|\nabla(V_h^{n+1} - V_h^{m+1})\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.13}$$

Now, since we assume that $\mathbf{u}$ is sufficiently regular, we have

$$|\mathbf{u}^{n+1} - \mathbf{u}^{m+1}| \leqslant |n-m|\Delta t \|\mathbf{u}'\|_{L^\infty(0,T_0)} \leqslant N\Delta t \|\mathbf{u}'\|_{L^\infty(0,T_0)} \leqslant T_0 \|\mathbf{u}'\|_{L^\infty(0,T_0)}. \tag{2.14}$$

So the additional term intervening can be bounded as follows:

$$\begin{aligned}&|(f(H_\beta(V_h^{n+1}))(\mathbf{u}^{n+1} - \mathbf{u}^{m+1}), \nabla(V_h^{n+1} - V_h^{m+1})|\\ &\quad \leqslant \tfrac{1}{4}\|\nabla(V_h^{n+1} - V_h^{m+1})\|^2\\ &\qquad + C(T_0)\|f(\cdot)\|^2_{L^2} \|\mathbf{u}'\|^2_{L^\infty(L^\infty)}.\end{aligned} \tag{2.15}$$

Finally combine (2.15) with [8, pp. 373–374], to get the theorem. □

So scheme (2.9)–(2.10) is well-defined, at least for Δt sufficiently small, even for $\mathbf{u}$ a function of time t. Notice that, in the case $\mathbf{u}$ is independent of t, the last term on the right-hand side of (2.12) vanishes, and we get the same result as in [8].

Since scheme (2.9)–(2.10) is nonlinear, there is need of some form of linearization. The usually proposed linearization is a Picard iteration [2,10]:

$$\left(\frac{H_\beta(V_h^{n+1,m}) - H_\beta(V_h^{n,m})}{\Delta t}, \chi\right) - (f(H_\beta(V_h^{n+1,m}))\mathbf{u}^{n+1}, \nabla\chi) + (\nabla V_h^{n+1,m+1}, \nabla\chi) = 0, \tag{2.16}$$

for all $\chi \in M_h$, $n = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M(n)$, with $M(n)$ sufficiently large, and with

$$P_h H_\beta V_h^0 = P_h S^0. \tag{2.17}$$

In the next section we provide different, simpler approach.

3. A linearization of the backward scheme

We propose here an alternative approach to linearize the backward scheme above. We assume the functions f and K, and hence K_β and H_β, are twice continuously differentiable. Thus we can write:

$$(f \circ H_\beta)(s) = (f \circ H_\beta)(s_0) + (s - s_0)(f \circ H_\beta)'(s_0) + O((s - s_0)^2), \tag{3.1}$$

$$H_\beta(s) = H_\beta(s_0) + (s - s_0)H_\beta'(s_0) + O((s - s_0)^2), \tag{3.2}$$

using Taylor expansions of order 1. Neglecting the big-O terms in the above expansions, scheme (2.9)–(2.10) then becomes

$$\begin{aligned}
&\left(\frac{U_h^{n+1} - U_h^n}{\Delta t} H_\beta'(U_h^n), \chi\right) \\
&\quad - (\{f \circ H_\beta(U_h^n) + (U_h^{n+1} - U_h^n)(f \circ H_\beta)'(U_h^n)\}\mathbf{u}^{n+1}, \nabla\chi) \\
&\quad + (\nabla U_h^{n+1}, \nabla\chi) = 0, \quad \forall \chi \in M_h, \\
&\quad P_h H_\beta U_h^0 = P_h S^0.
\end{aligned} \tag{3.3}$$

This is now a linear scheme. If $(e_i)_{1 \leqslant i \leqslant m}$ is a basis for M_h, and if

$$U_h^n = \sum_{1 \leqslant i \leqslant m} \alpha_i^n e_i,$$

then we get, at the time step $n+1$, after setting $\chi = e_j$,

$$\left(\sum_{1\leqslant i\leqslant m} \alpha_i^{n+1} H'_\beta(U_h^n)e_i, e_j\right) - \Delta t\left(\sum_{1\leqslant i\leqslant m} \alpha_i^{n+1}(f\circ H_\beta)'(U_h^n)\mathbf{u}^{n+1}e_i, \nabla e_j\right)$$

$$+\,\Delta t\left(\sum_{1\leqslant i\leqslant m} \alpha_i^{n+1}\nabla e_i, \nabla e_j\right) = B_j^n, \tag{3.4}$$

where the previously computed terms are regrouped in B_j^n. We get the system

$$\mathbf{A}\alpha = \mathbf{B} \tag{3.5}$$

with the $m\times m$ matrices $\mathbf{A}=\mathbf{A}(n)$ and $\mathbf{B}=\mathbf{B}(n)$, given by

$$\mathbf{A} = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3, \tag{3.6}$$

$$\mathbf{B} = \left(B_j^n\right)_{1\leqslant j\leqslant m}, \tag{3.7}$$

$$\boldsymbol{\alpha} = (\alpha_i^{n+1})_{1\leqslant i\leqslant m} \tag{3.8}$$

with

$$C_1(i,j) = (\lambda e_i, e_j),$$

$$C_2(i,j) = \Delta t(\mathbf{w}e_i, \nabla e_j),$$

$$C_3(i,j) = \Delta t(\nabla e_i, \nabla e_j), \tag{3.9}$$

where, for a fixed n, $\lambda = H'_\beta(U_h^n)$, and $\mathbf{w} = -(f\circ H_\beta)'(U_h^n)\mathbf{u}^{n+1}$.

One notices that the assembled matrix obtained is not symmetric which is expected because of the presence of the transport term. The following lemma will help in the proof of the main result of this paper (Theorem 3.1).

Lemma 3.1. *Under the conditions on k, f, and the definition* (*see* (2.2)) *of k_β, we have the following*.

$$\left|\frac{f'(s)}{k_\beta(s)}\right| \leqslant C\frac{1}{\sqrt{\delta}} \quad \forall s\in[0,1]. \tag{3.10}$$

Proof. We have

$$\left|\frac{f'(s)}{k_\beta(s)}\right| = \left|\frac{f'(s)}{\sqrt{k_\beta(s)}}\frac{1}{\sqrt{k_\beta(s)}}\right| \leqslant \left|\frac{f'(s)}{\sqrt{k_\beta(s)}}\frac{c}{\sqrt{\delta}}\right|, \tag{3.11}$$

where we have used (2.2). Next using (1.5), we get the lemma. □

Our main result is the following:

Theorem 3.1. *Let $v \in M_h$, and the matrix* **A** *be defined by* (3.9), *then*

$$(v^t, \mathbf{A}v) \geqslant c_2\left(1 - \frac{\sqrt{\Delta t}\, c_3(\mathbf{u})}{2}\right)\|v\|^2_{L^2(\Omega)} + \Delta t\|\nabla v\|^2_{L^2(\Omega)}. \tag{3.12}$$

Remark 3.1. This Theorem states that the matrix $\mathbf{A}(n)$, obtained at each time step n, is definite positive for Δt small. Hence the proposed linearized scheme is well defined, at least for Δt sufficiently small. Also notice that, in addition to being independent of β, h and Δt, c_2 and c_3 are independent of n, the time step at which the equation is considered. So (3.12) holds uniformly in n.

Proof. Let $m = \dim(M_h)$, and $v = (v_i)_{i=1}^m$, with respect to a basis $(e_i)_{i=1}^m$ of M_h. Set $\mathbf{A} = (a_{ij})$.

We have

$$\begin{aligned}\int_\Omega v^t A v \,\mathrm{d}x = (v^t, Av) = &\sum_{1\leqslant i\leqslant m}\left(v_i, \sum_{1\leqslant j\leqslant m} a_{ij}v_j\right)\\ &\times \sum_{1\leqslant i\leqslant m}\sum_{1\leqslant j\leqslant m}\left\{v_i(\lambda e_i, e_j)v_j + \Delta t v_i(\mathbf{w}e_i, \nabla e_j)v_j\right.\\ &\left. + \Delta t v_i(\nabla e_i, \nabla e_j)v_j\right\}.\end{aligned} \tag{3.13}$$

The above equality yields

$$\begin{aligned}\int_\Omega v^t A v\,\mathrm{d}x = &\left(\lambda\sum_{1\leqslant i\leqslant m} v_i e_i, \sum_{1\leqslant j\leqslant m} v_j e_j\right) + \Delta t\left(\sum_{1\leqslant i\leqslant m} v_i e_i \mathbf{w}, \sum_{1\leqslant j\leqslant m} v_j\nabla e_j\right)\\ &+\Delta t\left(\sum_{1\leqslant i\leqslant m} v_i\nabla e_i, \sum_{1\leqslant i\leqslant m} v_i\nabla e_i\right)\\ = &(\lambda v, v) + \Delta t(\mathbf{w}v, \nabla v) + \Delta t\|\nabla v\|^2_{L^2(\Omega)}\\ = &\|\sqrt{\lambda}v\|^2_{L^2(\Omega)} + \Delta t(\mathbf{w}v, \nabla v) + \Delta t\|\nabla v\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.14}$$

By the definition of λ and $\mathbf{w}$, setting $S_h^n = H_\beta(U_h^n)$, we get

$$\lambda = H'_\beta(U_h^n) = \frac{1}{k_\beta(H_\beta(U_h^n))} = \frac{1}{k_\beta(S_h^n)} \geqslant 0, \tag{3.15}$$

using definition (2.6) of H_β and (2.4). We also have

$$\mathbf{w} = -(f \circ H_\beta)'(U_h^n)\mathbf{u}^{n+1} = -f'(S_h^n)\frac{1}{k_\beta(S_h^n)}\mathbf{u}^{n+1}. \tag{3.16}$$

Using the Cauchy–Schwartz Inequality in (3.14), we get

$$\begin{aligned}\int_\Omega v^t A v \,\mathrm{d}x &\geqslant \|\sqrt{\lambda} v\|^2_{L^2(\Omega)} - \frac{\Delta t}{2}\{\|\mathbf{w} v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\} \\ &\quad + \Delta t \|\nabla v\|^2_{L^2(\Omega)} \\ &= \|\sqrt{\lambda} v\|^2_{L^2(\Omega)} - \frac{\Delta t}{2}\|\mathbf{w} v\|^2_{L^2(\Omega)} + \frac{\Delta t}{2}\|\nabla v\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.17}$$

Next, using (3.15) and (3.16), we get

$$\begin{aligned}\int_\Omega v^t A v \,\mathrm{d}x &\geqslant \left\| \frac{1}{\sqrt{k_\beta(S_h^n)}} v \right\|^2_{L^2(\Omega)} - \frac{\Delta t}{2} \left\| \frac{f'(S_h^n)}{k_\beta(S_h^n)} v \mathbf{u}^{n+1} \right\|^2_{L^2(\Omega)} \\ &\quad + \frac{\Delta t}{2}\|\nabla v\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.18}$$

Now we can choose δ and Δt such that

$$\frac{\sqrt{\Delta t}}{\sqrt{\delta}} \leqslant c_0, \tag{3.19}$$

for some positive constant c_0. We also have

$$\left| \frac{1}{k_\beta(S_h^n)} \right| \geqslant \frac{1}{\|k(\cdot)\|_\infty}. \tag{3.20}$$

Putting together (3.10), (3.18)–(3.20), we get

$$\begin{aligned}\int_\Omega v^t A v \,\mathrm{d}x &\geqslant \frac{1}{\|k(\cdot)\|_\infty} \|v\|^2_{L^2(\Omega)} - c_1 \frac{\sqrt{\Delta t}}{2} \|\mathbf{u}\|^2_{L^\infty(L^\infty)} \|v\|^2_{L^2} \\ &\quad + \frac{\Delta t}{2}\|\nabla v\|^2_{L^2}. \end{aligned} \tag{3.21}$$

The last inequality clearly yields the theorem. □

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