

RESEARCH SUMMARY

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1. Introduction

My research program has focused on investigating mainly equations obtained from modeling multi-phase flow through porous media. Example applications include groundwater flow and pollution (single phase flow, concentration problem), and oil reservoir simulations (immiscible flow, saturation problems[18]–[20]), but the techniques developed can be applied to other type of advection-diffusion equations. In addition to the strong coupling and nonlinearity, a main difficulty we address is the degenerate nature of many of these problems, that is the diffusion coefficient vanishes for some values of the solutions. In this case sufficient regularity of the solution is not guaranteed for numerical treatment [1][2]. This calls for some form of regularization. For the case of two-phase immiscible displacement through porous media, we have (with some simplification) the following nonlinear coupled system:

$$(1.1) \quad \left\{ \begin{array}{ll} \mathbf{u} = -a(s)\nabla p & \Omega \times (0, T] \\ \nabla \cdot \mathbf{u} = Q_1 & \\ \mathbf{u} \cdot \boldsymbol{\eta} = 0 & \partial\Omega \times [0, T] \\ \int_{\Omega} p dx = 0 & \forall t \in [0, T] \text{ (for uniqueness) .} \\ \frac{\partial S}{\partial t} + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S) & \\ (k(S)\nabla S) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times [0, T_0] \\ S(x, 0) = S^0(x) \geq 0 & \text{on } \Omega \end{array} \right.$$

The main goal is to approximate accurately and efficiently the numerical solution (p, S) of this system, where p is the global pressure of the phases, and S the saturation of the invading fluid, or (\mathbf{u}, S) , where \mathbf{u} is the Darcy velocity of the flow.

2. Completed work.

In our Ph.D Dissertation [6] (under the supervision of Dr. R. C. Sharpley, University of South Carolina, Columbia), we had considered the saturation equation, with the absence of the pressure part:

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$$(2.1) \quad \frac{\partial S}{\partial t} + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S)$$

with the Neuman type boundary condition

$$(2.2) \quad (f(S)\mathbf{u} - k(S)\nabla S) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times [0, T_0]$$

and the initial condition

$$(2.3) \quad 1 \geq S(x, 0) = S^0(x) \geq 0 \quad \text{on } \Omega$$

where Ω is a bounded domain in \mathbb{R}^n , with $n \leq 3$.

We regularized this general degenerate problem to a nondegenerate one, following the general direction as in [1]–[3]. With some control on how the diffusion coefficient vanishes, we established a priori estimates and error estimates for the regularized problem[7], and error estimates for a Galerkin finite element approximation of the problem[8]. Two papers have come out of this work[7][8]. After the dissertation a third paper[9] followed which gave additional error estimates not given in the dissertation[9].

Many authors have dealt with these types of problems, mainly with the nondegenerate case. In [10], we have considered the full system (1.1), and under reasonable physical assumptions on the data, we prove existence and uniqueness for the system. This was done independently of [12], which uses a different approach. In [11], we consider a standard Galerkin method applied to the full system (instead of a mixed finite element method for the pressure part and standard Galerkin method for the saturation as in [13]), and perform error analysis for both the continuous and the discrete methods.

The work done is still theoretical. In [8,9], some schemes are given but are not directly usable on a computation, since these schemes are nonlinear. We would need to linearize in some way. That is the work we have started in [2]1

3. Perspectives and Other Directions

3.1 Perspectives. The future focus of my research program, will be on effective numerical treatment of the solution to problem (1.1) using efficient numerical methods, the most commonly proposed being a combination of a mixed finite element method and a standard finite element method with or without a prior regularization[13]–[15]. In a forthcoming paper(see [21]), we propose a numerical method linearizing a nonlinear Euler backward method for the saturation equation. Since we are concerned with accurately approximating the solution, this mixture (mixed and standard finite element methods) will yield better results than applying standard finite element methods to both the pressure and the saturation parts of (1.1). Another promising method is the Eulerian Lagrangian Localized Adjoint Method (ELLAM) [4]–[5]. This method has been well developed for linear advection-diffusion equations[16]–[17]. We will study a hybrid method which applies ELLAM to the saturation equation (2.1) and the mixed finite element method to the pressure. Unfortunately, this is a nonlinear advection-diffusion equation and there are several possibilities to consider, for example:

- Linearize (2.1), by the Picard iteration, for instance;
- Develop the ELLAM theory for nonlinear equations;

The last consideration is a joint research topic with R. Ewing, R. Sharpley, and H. Wang at the University of South Carolina, Columbia, and the Institute for Scientific Computation, Texas A & M University, College Station.

3.2 Other Directions and subjects of interest.

- Population Dynamics Equations
- Atmospheric Dynamics.
- More general PDE (linear or nonlinear): How do they work, for instance, with *Wavelets*?

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