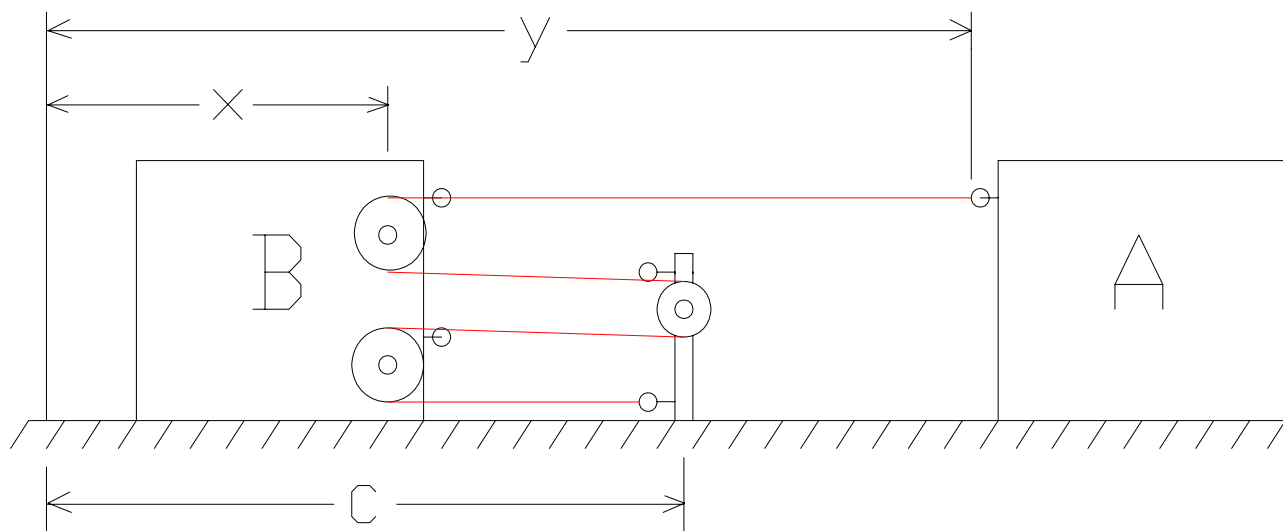


**11th ANNUAL VALDOSTA STATE UNIVERSITY
MATHEMATICS TECHNOLOGY CONFERENCE
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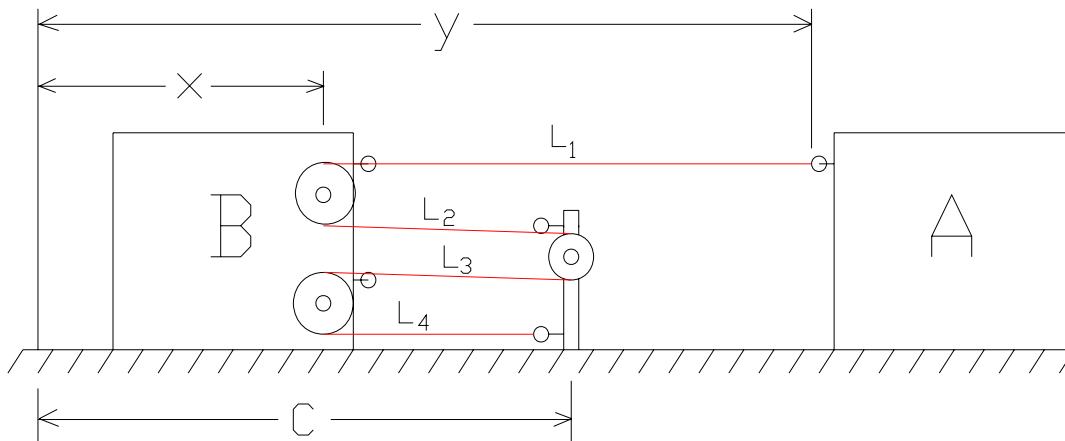
**MATHEMATICAL
MODELING OF SIMPLE
MECHANICAL DEVICES**



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As one manually moves blocks **A** and **B**, it is observed that both blocks move in the same direction and that the distance between the blocks is changing - either increasing or decreasing, depending upon the direction of motion. Most students will not recognize an essential fact which makes it possible for us to develop a mathematical model for the relative of the two blocks. The essential fact is that, although the distance between the blocks is changing, **the length of the cable which connects the blocks is NOT changing**. As shown in the diagram below, we can express the length of the cable as viewed from any arbitrary (and **FIXED**) point. In the diagram below, the arbitrary point selected is at **P**, a fixed distance to the left of the post which contains one of the system pulleys.



We can write a linear equation which models the constant length of the cable as follows:

$$\mathbf{L_1 + L_2 + L_3 + L_4 = \text{CONSTANT}} \quad [1]$$

We also observe that the various lengths of cable may be represented as:

$$\mathbf{L_1 = y - x} \quad \mathbf{L_2 = C - x} \quad \mathbf{L_3 = C - x} \quad \mathbf{L_4 = C - x} \quad [2]$$

So the constant length of the cable may be written as:

$$\mathbf{(y - x) + (C - x) + (C - x) + (C - x) = \text{CONSTANT}} \quad [3]$$

It is important to point out here that some of the cable length is wrapped about the pulleys and so is not included in the length of the cable expressed by equations, [1], [2], and [3]. The length of cable which is wrapped about the pulleys does not change (is constant) as the blocks move. That is, the constant length of cable in contact with the pulleys can be combined with the constant which appears on the right side of equation [3] to produce a new constant. The important issue in this exercise is the changing positions of the blocks. The constant terms are of little concern.

Combining some terms in equation [3]:

$$\mathbf{(y - x) + 3(C - x) = \text{CONSTANT}}$$

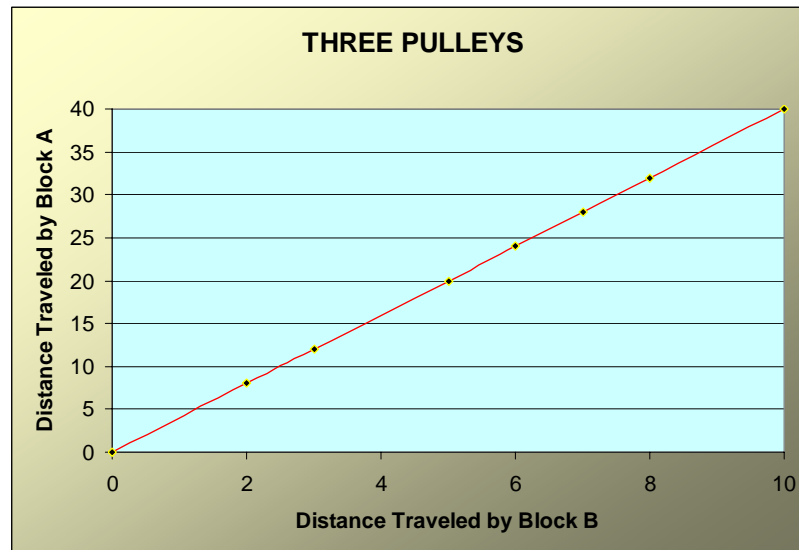
$$y - 4x + 3C = \text{CONSTANT} \quad [4]$$

Noting that the distance C is also constant, we can combine the 3C term of equation [4] with the CONSTANT term on the right side of the equation to get:

$$y - 4x = \text{CONSTANT}$$

or
$$y = 4x + \text{CONSTANT} \quad [5]$$

We get a linear equation which has a slope of 4. The y-intercept will be the constant length of the cable, but we realize that what is being measured with the apparatus is the **change in position** of blocks **A** and **B**. These changes in position will have a ratio of 4 units for block **A** to one unit for block **B**. There will be, of course, no change in the length of the cable. Graphing the changes in position for the blocks, we get the results illustrated below (note the zero y-intercept, which merely indicates that if block B does not move neither does block A move).



It is important to consider the signs in equation [5]. Because all terms are positive, the equation tells us that the blocks will move in the same direction, either left or right. Recall that distances were measured from a fixed point, P. The equation is telling us that, if one of the dimension from point P increases, the other dimension must also increase (and vice versa).

We can consider the rates of change of the block positions by taking the time derivative of equation [5].

$$\frac{dy}{dt} = 4 \frac{dx}{dt} + 0 \quad [6]$$

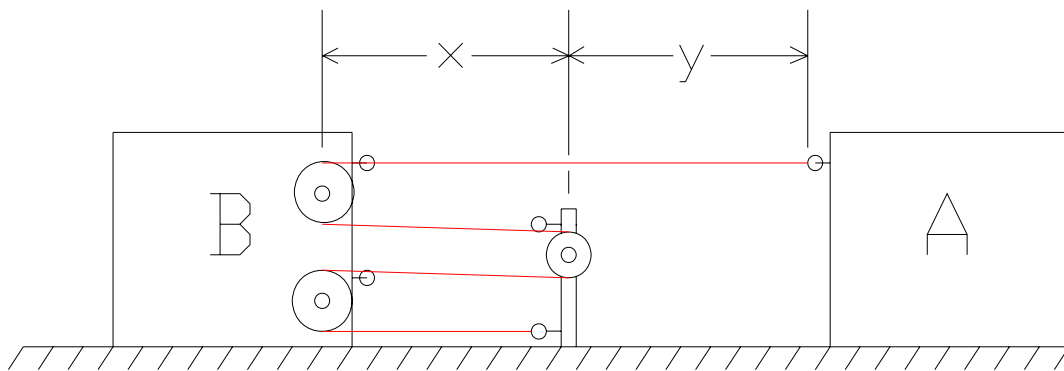
Equation [6] tells us that block B will move four times as fast as block, regardless of direction. Note that when block A begins to move, block B also begins to move. Similarly, the blocks stop together, so the period of time during which they in motion **MUST** be identical for each block - Δt is the same for both blocks.

Accelerations can be modeled by taking the time derivative of equation [6].

$$\frac{d^2 y}{dt^2} = 4 \frac{d^2 x}{dt^2}$$

We see that, if block A is accelerating to the right, block B will be accelerating four times as fast in the same direction.

An interesting case arises when distances are measured from a point between the two blocks, such as from the post which contains a single pulley. The diagram then looks like:



The length of the cable may be modeled as:

$$\mathbf{y + 4x = \text{CONSTANT}}$$

or

$$\mathbf{y = -4x + \text{CONSTANT}} \quad [7]$$

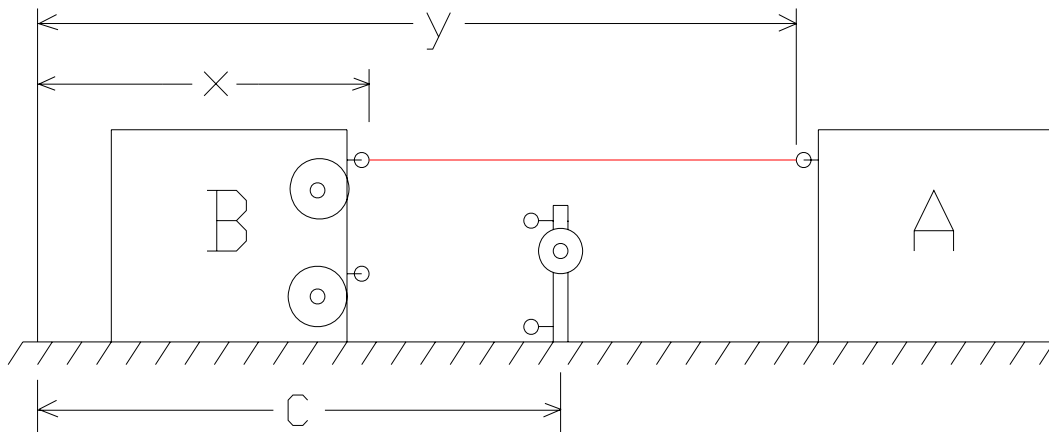
Notice that in this case a negative sign appears on the coefficient of x. How is this case different from the earlier case? The negative sign indicates that, as one dimension (say y) increases, then the other dimension must decrease. Taking the time derivative of equation [7], we get:

$$\frac{dy}{dt} = -4 \frac{dx}{dt} \quad [8]$$

which tells us that as one block moves away from the center post, the other block must be moving toward the center post. Taking the second derivative with respect time, we see that the same must be true of acceleration.

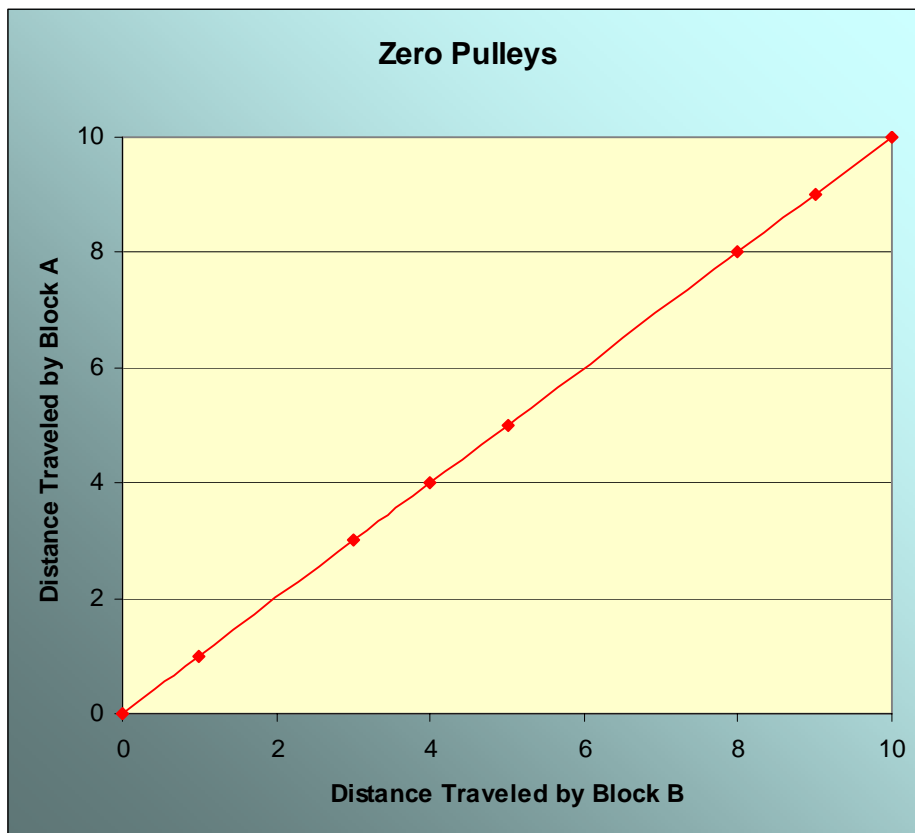
$$\frac{d^2 y}{dt^2} = -4 \frac{d^2 x}{dt^2} \quad [9]$$

SYSTEM SETUP USING NO PULLEYS



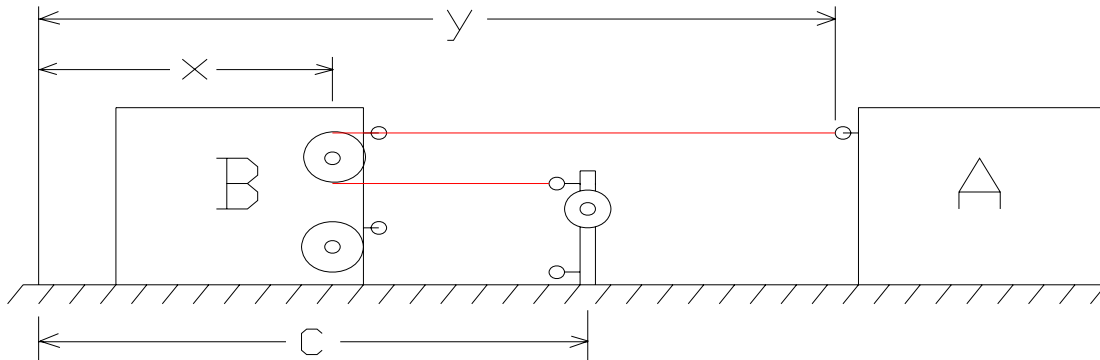
MATH MODEL: $Y - X = \text{CONSTANT}$

DISTANCES TRAVELED BY BLOCKS A AND B								
BLOCK A (Y-Distance)	0	1	3	4	5	8	9	10
BLOCK B (X-Distance)	0	1	3	4	5	8	9	10



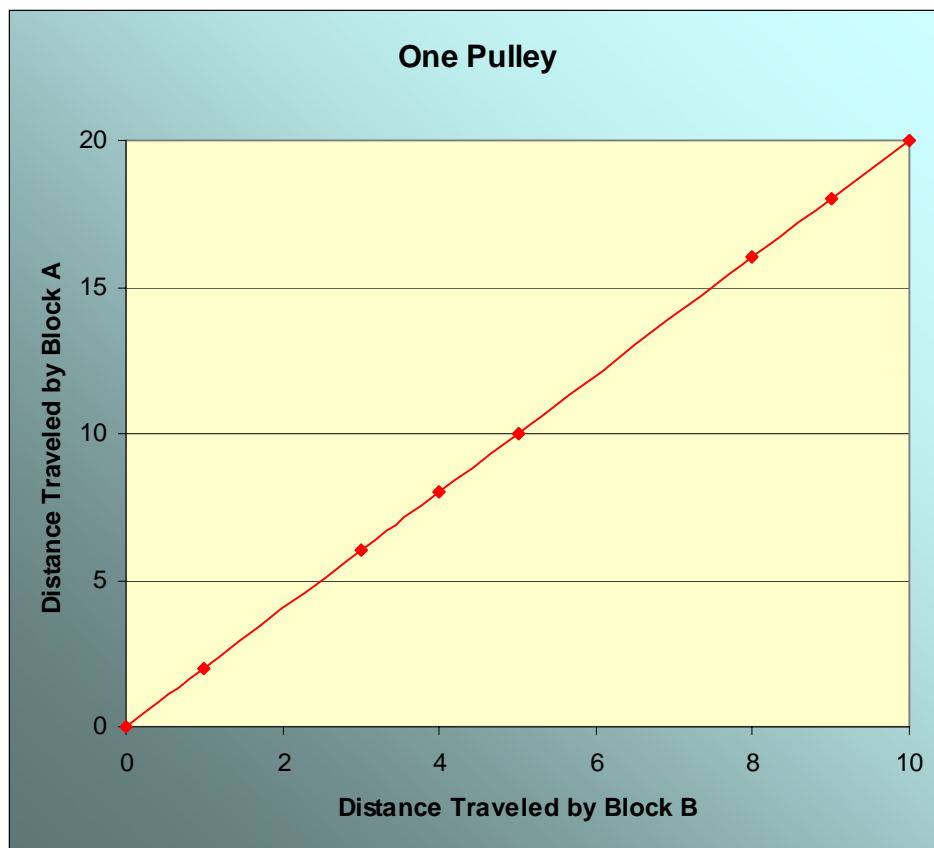
We see that the slope of the line is 1.

SYSTEM SETUP USING ONE PULLEY



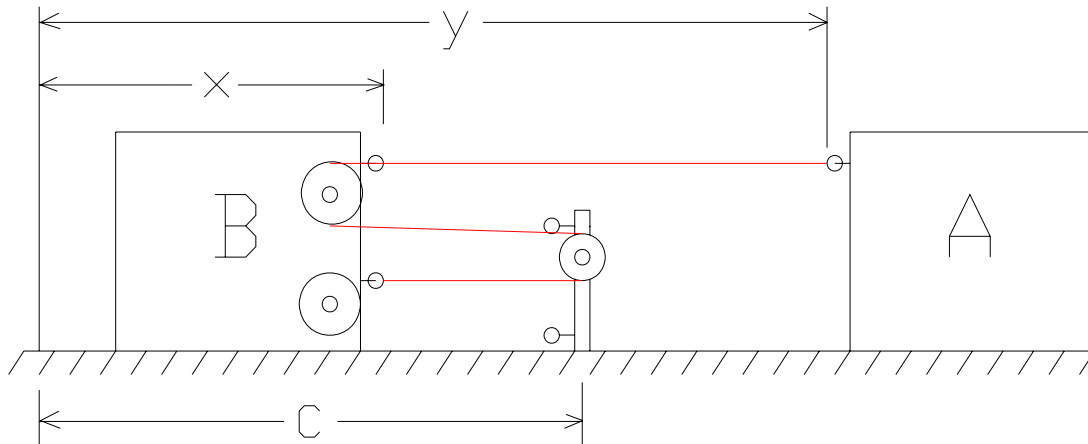
MATH MODEL: $Y - 2X = \text{CONSTANT}$

DISTANCES TRAVELED BY BLOCKS A AND B								
BLOCK A (Y-Distance)	0	2	4	6	10	14	16	20
BLOCK B (X-Distance)	0	1	2	3	5	7	8	10



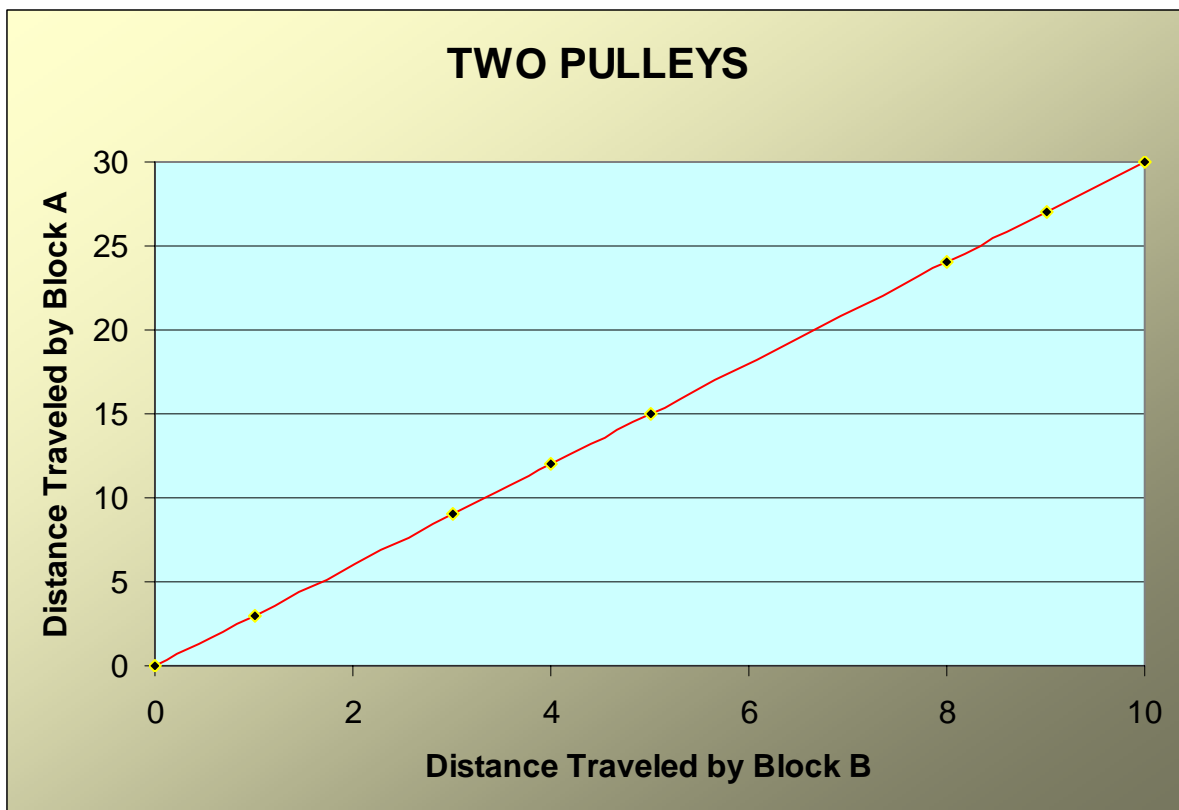
We see that the slope of the line is 2.

SYSTEM SETUP USING TWO PULLEYS



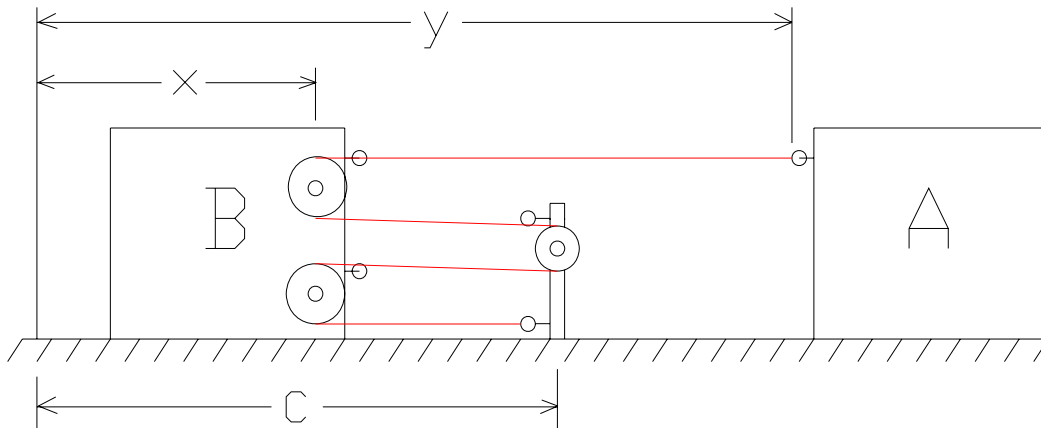
MATH MODEL: $Y - 3X = \text{CONSTANT}$

DISTANCES TRAVELED BY BLOCKS A AND B								
BLOCK A (Y-Distance)	0	3	9	12	15	24	27	30
BLOCK B (X-Distance)	0	1	3	4	5	8	9	10



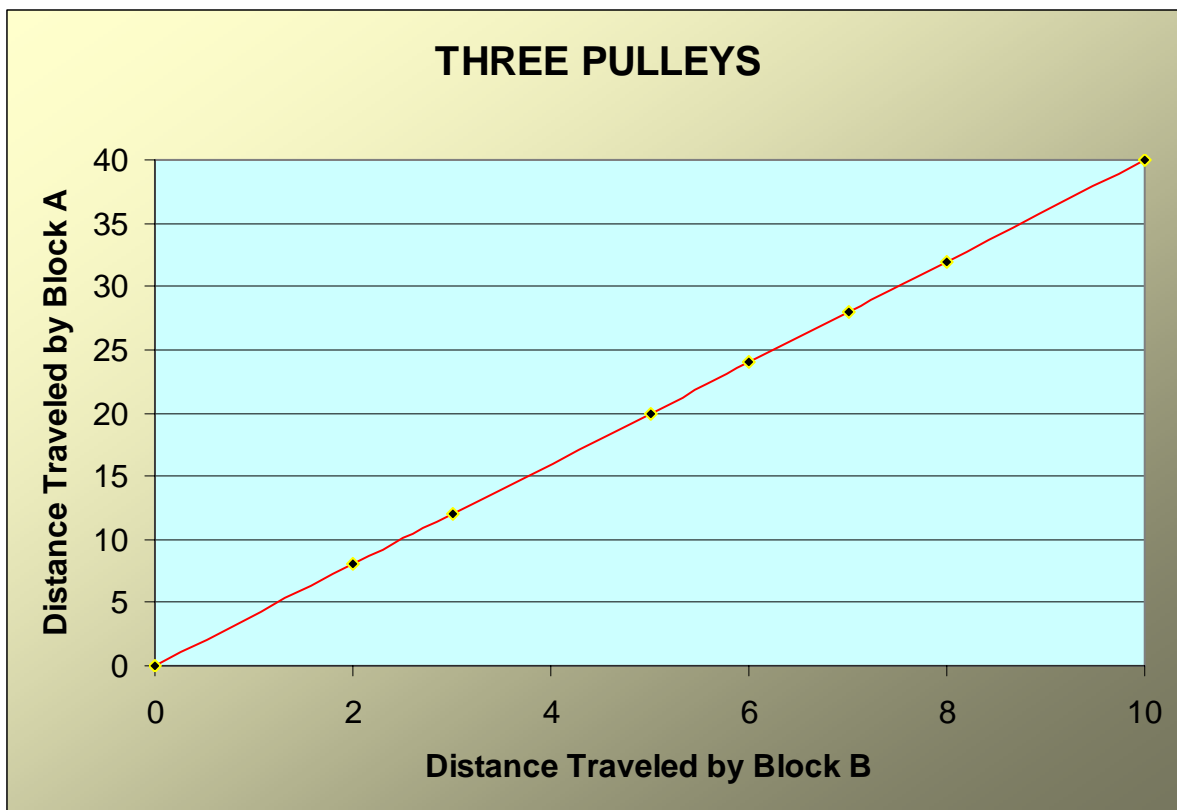
We see that the slope of the line is 3.

SYSTEM SETUP USING THREE PULLEYS



MATH MODEL: $Y - 4X = \text{CONSTANT}$

DISTANCES TRAVELED BY BLOCKS A AND B								
BLOCK A (Y-Distance)	0	8	12	20	24	28	32	40
BLOCK B (X-Distance)	0	2	3	5	6	7	8	10



We see that the slope of the line is 4.

SUPERIMPOSING THE RESULTS

As a final demonstration, the graphs of all four trials are superimposed on a single graph. The point here is to emphasize to students that the scales used in preparing a graph are very important. As each trial was done, the resulting graphs appeared to have the same (or very similar) slopes. This appearance is due to the selection of the vertical scale - 1-10 for no pulleys, 1-40 for three pulleys). It is only when the graphs of the four trials are superimposed that the difference in the slopes of the graphical results are appreciated.

